Math 2010 B Tutorial 11

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Outline:

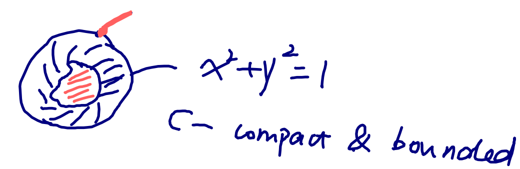
· Optimization problem on a region which is <u>not compact</u>.
(i.e not both closed & bounded)

e.g. Let $f(x,y) = \dfrac{1}{x^2+4y^2}$

Find the global minimum of f (if it exists) on the region $\underline{R} = \{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 \leq 1\}$.

R is bounded. but not closed.

Sol: Observe the nature of the region on which f is to be minimized



☆ strategy: Find

compact ⟶ EVT ⟶ mininum on C

① a closed and bounded subset C of $\underline{R}$; and

② a suitable $\underline{p}$ $(t_0, x_0) \in \underline{C}$ (for comparision)

s.t. $f(t,x) \geq \underline{f(t_0, x_0)}$ $\forall (t,x) \in R \setminus C$ if possible.

$\Rightarrow$ $\underline{\text{minimum of } f \text{ on } R}$ exist. moreover $\underline{= \text{minimum of } f \text{ on } C}$.

back to e.g.

claim: $\exists\, r \in (0,1)$ s.t. $\forall\, x,y \in R$ w/ $x^2+y^2 \le r^2$,

$$f(x,y) \ge f(1,0) = 1$$

Then $C := \{(x,y) \in R : x^2+y^2 \ge r^2\} = \{(x,y) \in \mathbb{R}^2,\ \overbrace{1 \ge x^2+y^2 \ge r^2}^{\text{compact}}\}$

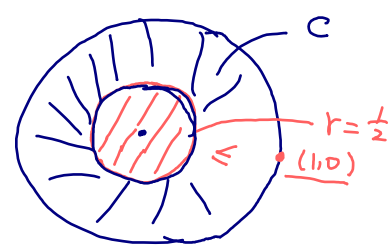
By EVT, f attains a min on C, which is also a minimum on R.
by the above claim.

[proof of claim for e.g.]

let $r = \frac{1}{2}$ Fix $(x,y) \in R$ w/ $x^2+y^2 \le r^2 = \frac{1}{4}$,

Then $x^2+4y^2 \le 4x^2+4y^2 = 4(x^2+y^2) \le 4\cdot r^2 = 1 \Rightarrow x^2+4y^2 \le 1 \Rightarrow \frac{1}{x^2+4y^2} \ge 1$

$\therefore f(x,y) = \frac{1}{x^2+4y^2} \ge 1 = f(1,0) = \frac{1}{1^2}$.



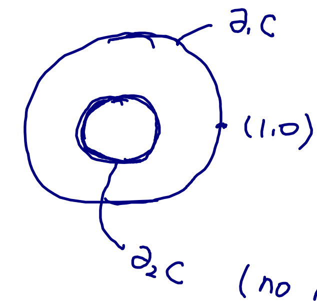
$\rightsquigarrow$ Find critical points of f in the interior of C.

Our problem is equi to find global max of

$$g(x,y) = x^2 + 4y^2 = \frac{1}{f(x,y)} \quad \text{on } C.$$

$g_x = 2x \;;\; g_y = 8y$

$\therefore$ g has n critical point on C



(no need to care about)

· For $\partial_1 C$

find global max of $g = x^2 + 4y^2$ on $\partial_1 C = \{ (x,y) \in \mathbb{R}^2, \; x^2+y^2=1 \}$

$$g = x^2 + 4y^2 = 1 + 3y^2 \qquad 0 \le y^2 \le 1$$

$\Rightarrow$ $g(x,y)$ has a global maximal at $(0,1)$

$$g(0,1) = 1 + 3 \times 1^2 = 4$$

rk: $g(x,y) = x^2 + 4y^2$

$= 1 + 3\sin^2\theta$

$\rightsquigarrow$ g has maximal at $\theta = \frac{\pi}{2}$.

$g(0,1) = 4$.

Conclusion: f has a global min at $(0,1)$ on R w/

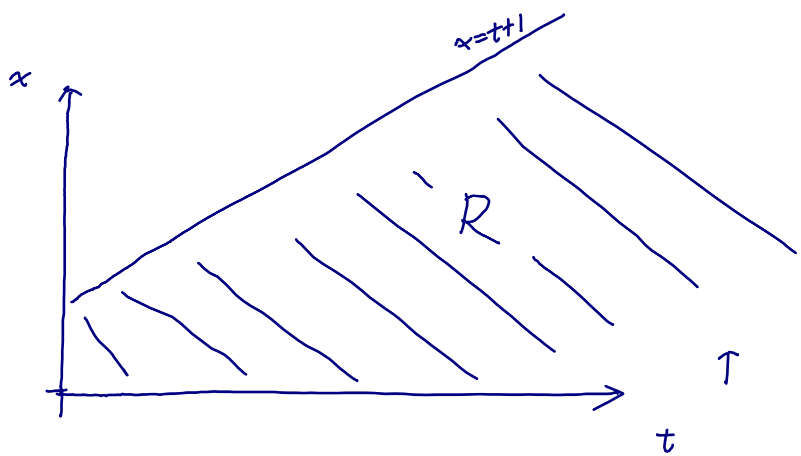
$$f(0,1) = \frac{1}{g(0,1)} = \frac{1}{4}.$$

e.g. Let $f(t,x) = (x^2 - 2x + t)\, e^{-t}$

Find the global maximum of f (if it exists). on the region

$$R = \{ (t,x) \in \mathbb{R}^2 : \underline{t \geq 0}, \ \underline{0 \leq x} \leq t+1 \}$$

Sol : Observe the nature of the region on which f is to be maximized



R is closed, but not bounded.

Strategy: Find

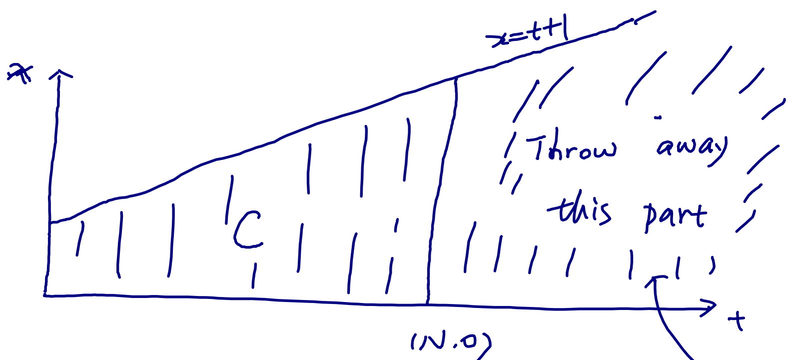
① a closed and bounded subset C of R; and

② a suitable pt $(t_0, x_0) \in C$ (for comparison)

s.t. $f(t,x) \boxed{\le} f(t_0, x_0)$ $\forall (t,x) \in R \setminus C$, if possible

Claim: $\exists N > 0$ s.t. $\forall (x,y) \in R$. w/ $t \geq N$

$$f(t,x) \le f(3,0) = e^{-3}.$$



$f(t,x) \le f(3,0)$.

Then $C := \{(t,x) \in R : t \le N\}$ is closed and bounded.

By EVT, f attains a max on C, which is also a maxi on R

by the above claim.

- proof of claim:

$\forall (t,x) \in R$,

$$|f(t,x)| = |(x-1)^2 + t - 1|\, e^{-t}$$

$$\le ((x-1)^2 + t + 1)\, e^{-t} \qquad \triangle \text{ inequ}$$

$$\le (t^2 + t + 1)\, e^{-t} \qquad (x \le t+1)$$

$$\lim_{t \to +\infty} (t^2+t+1)e^{-t} = \lim_{t\to+\infty} \frac{t^2+t+1}{e^t} = 0 \qquad (\text{L'Hopital's Rule on } \tfrac{\infty}{\infty} \text{ form})$$

Note $f(3,0) = e^{-3} > 0$ $\therefore \exists\, N > 0$ (in fact $N > 3$) s.t

$\forall (t,x) \in R$ w/ $t \ge N$, $f(t,x) \le |f(t,x)| \le (t^2 + t + 1)\, e^{-t} \le f(3,0)$.